Linear Algebra review

• Vector space: $V = \{ |a\rangle, |b\rangle, |c\rangle, \ldots, |v\rangle, |w\rangle, \ldots \}$

$|v\rangle$

- • closed under addition, $|a\rangle + |b\rangle \in V$
- • $|n\rangle = 0$ null vector $\quad |a\rangle + |n\rangle = |a\rangle$
- • closed under multiplication by a scalar.

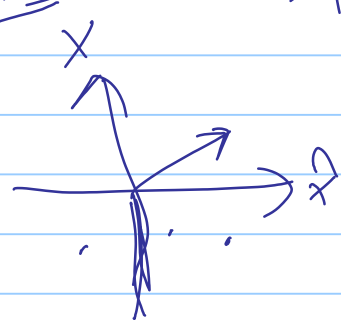
$$s|a\rangle \in V$$

$$\alpha|a\rangle + \beta|b\rangle \in V$$

Example: $i = \sqrt{-1}$

$\mathbb{C}^3$: $V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2+i \\ 3 \\ 4 \end{pmatrix}, \ldots, \begin{pmatrix} 2.1 \\ -1 \\ 0 \end{pmatrix}, \ldots \right\}$

$W = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2.1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2.1 \\ 2.1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \ldots \right\}$

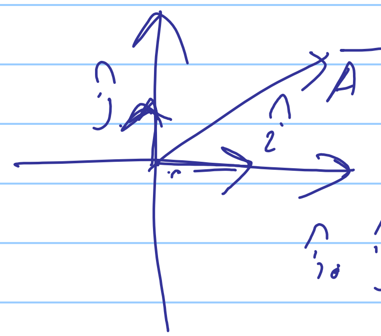


• **Basis** $\{|v_1\rangle, |v_2\rangle, |v_3\rangle \ldots\}$.

$$|V\rangle = \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle + \alpha_3 |v_3\rangle \ldots$$

$\{\hat{i}, \hat{j}\}$

$\vec{A} = 2\hat{i} + 3\hat{j}$



$\hat{i} \cdot \hat{j} = 0$

$\mathbb{C}^3$ $|v_1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|v_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $|v_3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$|V\rangle = \begin{pmatrix}2+i\\-i\\3\end{pmatrix} = (2+i)\begin{pmatrix}1\\0\\0\end{pmatrix} + (-i)\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle + \alpha_3 |v_3\rangle$$

$$|V\rangle = \sum_{i=1}^{3} \alpha_i |v_i\rangle$$

Inner product / dot product

$|v\rangle, |w\rangle \Longrightarrow \langle w|v\rangle$

$\langle w| = |w\rangle^{\dagger}$

(conj, transpose)

$$= (\langle w|)(|v\rangle)$$

$$|w\rangle = \begin{pmatrix}1\\-2i\\2\end{pmatrix} \Longrightarrow \langle w| = (1 \;\; 2i \;\; 2)$$

$$\langle w|v\rangle = (w_1^* \ w_2^* \ w_3^*)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= w_1^* v_1 + w_2^* v_2 + w_3^* v_3$$

Example

$$|v\rangle = \begin{pmatrix} -i \\ 3+i \\ 0 \end{pmatrix}, \quad |w\rangle = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$\langle v|w\rangle = (i \ \ 3-i \ \ 0)\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$$= 0 + 6 - 2i + 0 = 6 - 2i$$

$$\langle w|v\rangle = 6 + 2i = \langle v|w\rangle^*$$

$$= \langle v|v\rangle = (v_1^* \ v_2^* \ v_3^*)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= v_1^* v_1 + v_2^* v_2 + v_3^* v_3 = |v_1|^2 + |v_2|^2 + |v_3|^2$$

$$\geq 0$$

$$|v\rangle = \begin{pmatrix} 2i \\ 3-i \\ 1 \end{pmatrix} \quad \langle v|v\rangle = (-2i \ \ 3+i \ \ 1)\begin{pmatrix} 2i \\ 3-i \\ 1 \end{pmatrix}$$

$$= -4i^2 + 9 - i^2 + 1$$

$$= 4 + 10 + 1 = 15 \geq 0$$

$$|v'\rangle = \frac{1}{\sqrt{\langle v|v\rangle}}|v\rangle = \frac{1}{\sqrt{15}}\begin{pmatrix} 2i \\ 3i \\ 1 \end{pmatrix}$$

$$\boxed{\langle v'|v'\rangle = 1}$$

Normalized

Orthogonal

$\langle v|w\rangle = 0$ $|v\rangle$ & $|w\rangle$ are orthogonal

Orthonormal basis

$= \{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$

$$\langle v_i|v_j\rangle = \begin{cases} 0, & i \neq j \\ 1 & i = j \end{cases}$$

$\langle v_1|v_1\rangle = 1$, $\langle v_2|v_2\rangle = 1$, $\langle v_3|v_3\rangle = 1$

$\langle v_1|v_2\rangle = 0$, $\langle v_1|v_3\rangle = 0$ ...

Hilbert Space / Inner-product Space

— A vector space, with an inner product defined

— normalized vectors.

Linear operators:

$$|v\rangle = |v_1\rangle + |v_2\rangle + |v_3\rangle + \dots$$

$$|v\rangle = \sum_{i=1}^{n} |v_i\rangle$$

$$A|v\rangle = A|v_1\rangle + A|v_2\rangle + \dots$$

$$A|v\rangle = \sum_{i=1}^{n} A|v_i\rangle$$

Linear operator $A \equiv$ matrix

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$$A|v\rangle = \begin{pmatrix} \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix}$$

Examples

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2i \\ 1 \end{pmatrix}, \dots \right\}$$

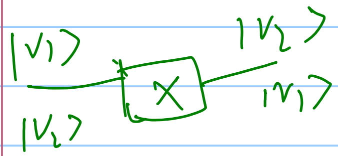
$$|v_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |v_2\rangle$$

$$\langle v_1 | v_2 \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= 0 + 0 = 0$$

$$|w_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} ; \quad |w_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\langle w_1 | w_2 \rangle = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$= \frac{1}{2}(1-1) = 0$$

Examples:



$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{X}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

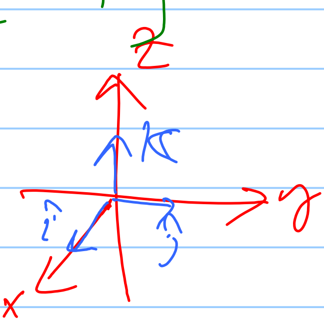
$$X|v_1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0+1 \\ 0+0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |v_2\rangle$$

$$X|v_2\rangle = |v_1\rangle$$

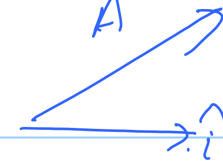
$$Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \underline{\underline{Z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Analogy:

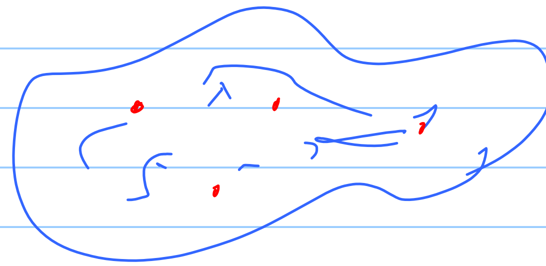
$$\bar{A} = \hat{i} A_i + \hat{j} A_j + \hat{k} A_k$$
$$= \sum_{m=i}^{k} \underline{\underline{\hat{m}}}\, \underline{\underline{A_m}}$$



$\hat{i} \cdot \vec{A} = A_x$



$\{|v_1\rangle, |v_2\rangle, \ldots\}$



$$|v\rangle = \sum_{i=1}^{n} \alpha_i |v_i\rangle$$

$$\alpha_i = \langle v_i | v\rangle$$

$$|v\rangle = \sum_{i=1}^{n} |v_i\rangle\langle v_i| v\rangle = \Big(\underbrace{\sum_{i=1}^{n} |v_i\rangle\langle v_i|}_{I}\Big)|v\rangle$$

$$I = \sum_i |v_i\rangle\langle v_i|$$

Outer product:

$$|v\rangle\langle w| = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} w_1^* & w_2^* & w_3^* \end{pmatrix}$$

$$= \begin{pmatrix} v_1 w_1^* & v_1 w_2^* & v_1 w_3^* \\ v_2 w_1^* & v_2 w_2^* & v_2 w_3^* \\ v_3 w_1^* & v_3 w_2^* & v_3 w_3^* \end{pmatrix}_{3\times 3}$$

Examples

$$|v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |w\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|v\rangle\langle w| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|v\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |w\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|v\rangle\langle w| = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$I = \sum_{i=0}^{1} |i\rangle\langle i|$$

Note Title 14/06/2021

Tensor product:

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} ; \quad |w\rangle = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$\langle v|w\rangle$ — inner product

$|w\rangle\langle v|$ — outer product

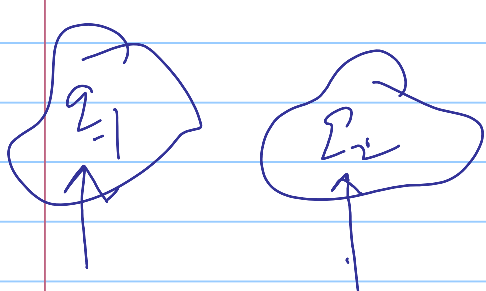
$$|v\rangle \otimes |w\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_1 \end{pmatrix}$$

$$\langle w| \otimes \langle w| = \begin{pmatrix} w_1^* & w_2^* \end{pmatrix} \otimes \begin{pmatrix} v_1^* & v_2^* \end{pmatrix}$$

$$= \begin{pmatrix} w_1^* v_1^* & w_1^* v_2^* & w_2^* v_1^* & w_2^* v_2^* \end{pmatrix}$$

$$X \otimes X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



$$= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$|0\rangle \otimes |1\rangle = |0\rangle|1\rangle = |01\rangle$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|1\rangle \otimes |1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

q_1 q_2

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = a\,|0\rangle|0\rangle + b\,|0\rangle|1\rangle + c\,|1\rangle|0\rangle + d\,|1\rangle|1\rangle$$