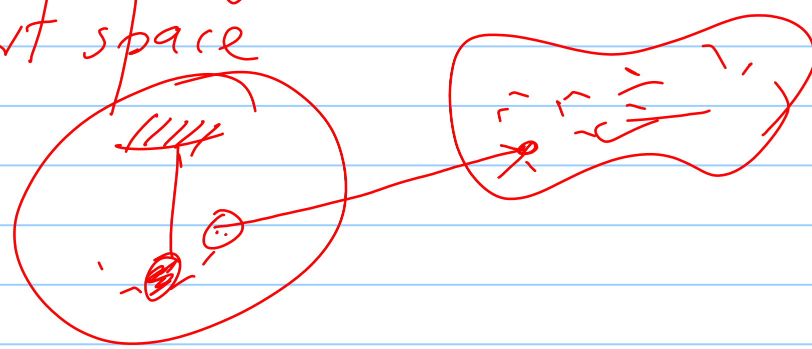
Introduction to Q.M.

Note Title 15/06/2021

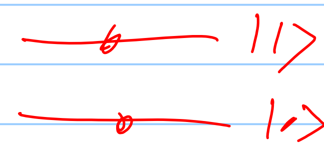
Examples: Two-level Q.S: Qubit

Postulate 1: State Space

→ With every system, assign a Hilbert space



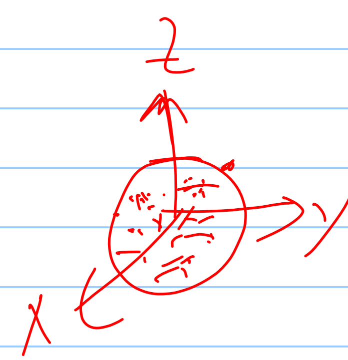
$$\underline{|\psi\rangle} = a|0\rangle + b|1\rangle$$



$$\boxed{|a|^2 + |b|^2 = 1}$$

$a = a' + i a''$; $b = b' + i b''$

$= r_1 e^{i\phi_1}$; $b = r_2 e^{i\phi_2}$



$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$= r_1 e^{i\phi_1}|0\rangle + r_2 e^{i\phi_2}|1\rangle$$

$$= e^{i\phi_1}\left[r_1|0\rangle + r_2 e^{i(\phi_2-\phi_1)}|1\rangle \right]$$

$$\boxed{r_1^2 + r_2^2 = 1} \quad ; \phi = \phi_2 - \phi_1$$

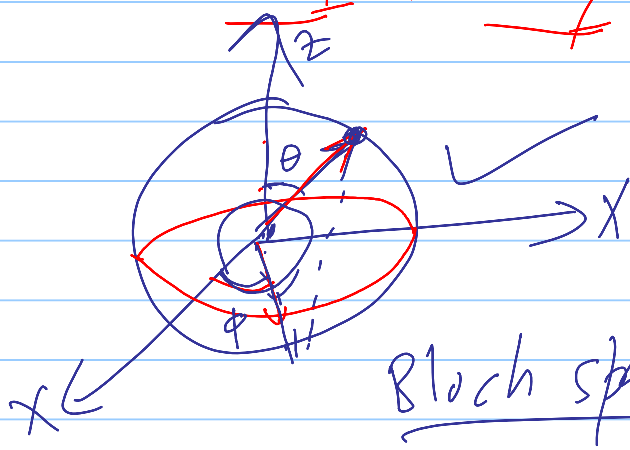
$$|\psi\rangle = e^{i\phi_1}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right]$$

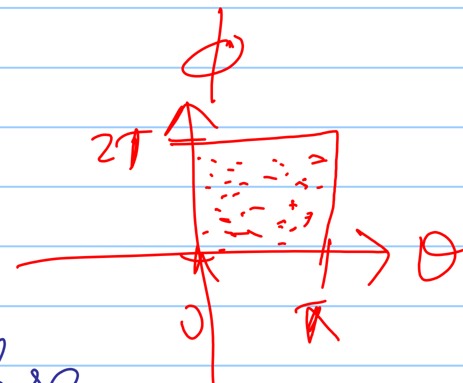
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$0 \le \phi \le 360^\circ = 2\pi$$

$$0 \le \frac{\theta}{2} \le 90^\circ \Rightarrow 0 \le \theta \le 180^\circ = \pi$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

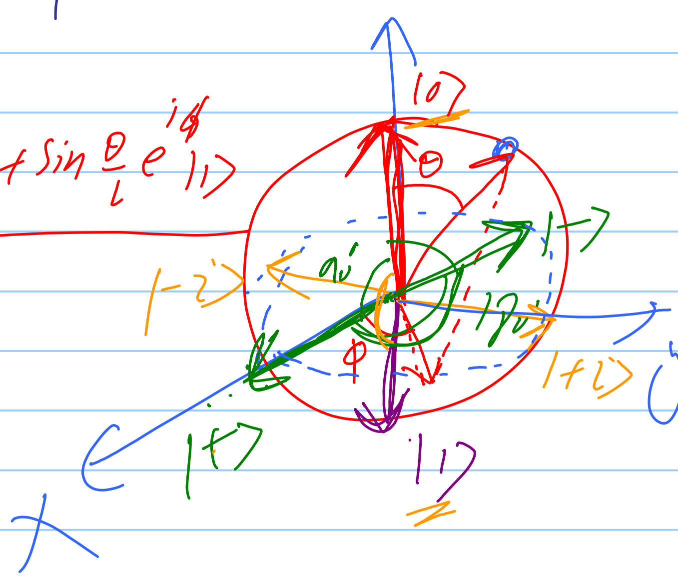




Bloch sphere

$$|\psi\rangle = |0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$$

$$\frac{\theta}{2} = 0^\circ \Rightarrow \boxed{\theta = 0^\circ}$$



$$|\psi\rangle = |1\rangle, \quad \theta = 180^\circ$$

$$\phi = \gamma$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\phi = 0^\circ, \quad \theta = 90^\circ$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) ; \quad \theta = 90^\circ$$

$$\phi = 180^\circ$$

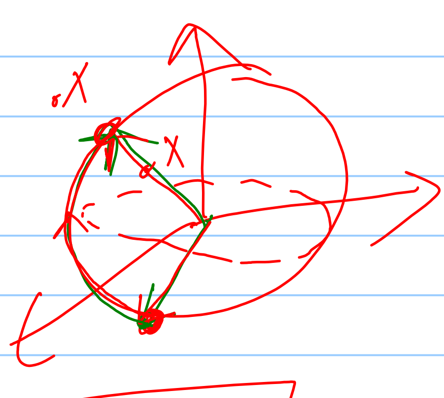
$$\boxed{\langle + | - \rangle = 0} \qquad \boxed{e^{i\pi} = -1}$$

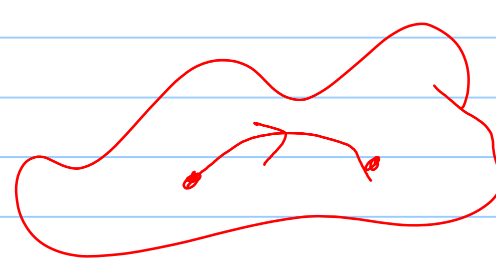
$$|\psi\rangle = \underbrace{\frac{i}{\sqrt{3}}}_{a}|0\rangle + \underbrace{\frac{(1-i)}{\sqrt{3}}}_{b}|1\rangle$$

$$|+i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

Postulate 2: Evolution / change of state





$$|\psi'\rangle = U|\psi\rangle$$

$$\boxed{\langle \psi | \psi \rangle = 1} \qquad \text{require: } \boxed{\langle \psi' | \psi' \rangle = 1}$$

if U is a unitary matrix

$$U^\dagger = (U^T)^* = U^{-1}$$

$$|\psi'\rangle = U|\psi\rangle$$

$$\langle\psi'| = (U|\psi\rangle)^\dagger = |\psi\rangle^\dagger U^\dagger$$

$$= \langle\psi|U^\dagger$$

$$\langle\psi'|\psi'\rangle = \langle\psi|\underbrace{U^\dagger U}|\psi\rangle$$

$$\langle\psi'|\psi'\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle$$

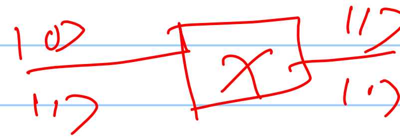
$$= 1$$

$$\boxed{U^\dagger = U^{-1}}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$X|0\rangle = |1\rangle$

$X|1\rangle = |0\rangle$



i	o
$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$

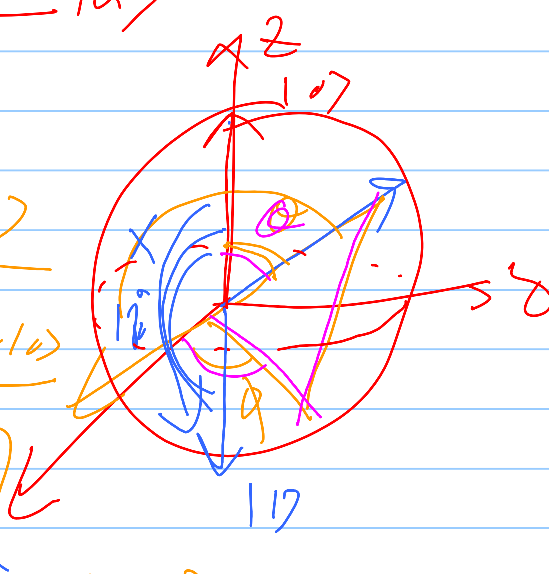
$$X^\dagger = (X^T)^* = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^{*T}$$

$$= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = X$$

$$X \cdot X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \cdot \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\boxed{X^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I}$$

$$|\alpha\rangle \;-\!\!\boxed{X}\!\!-\!\!\boxed{X}\!\!-\; |\alpha\rangle$$



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

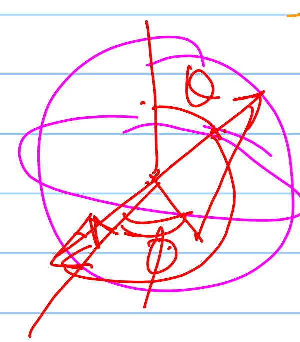
$$|\psi'\rangle = X|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle$$

$$= e^{i\phi}\left[\sin\frac{\theta}{2}|0\rangle + e^{-i\phi}\cos\frac{\theta}{2}|1\rangle\right]$$

$$= e^{i\phi}\left[\cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + e^{i(2\pi-\phi)}\sin\left(\frac{\theta+\pi}{2}\right)|1\rangle\right]$$

$$\theta \longrightarrow \theta + \pi$$

$$\phi \longrightarrow 2\pi - \phi$$



$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left(|0\rangle\langle 0| - |1\rangle\langle 1| \right)$$

$$Z^\dagger = Z, \quad \boxed{Z^2 = I}$$

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1| \right)|1\rangle$$

$$= |0\rangle\langle 0|1\rangle - |1\rangle\langle 1|1\rangle$$

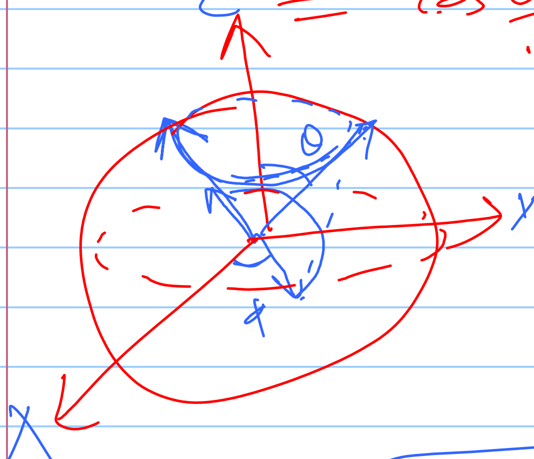
$$\boxed{Z|1\rangle = -|1\rangle}$$

$$= e^{i\pi}|1\rangle$$

$$Z|\psi\rangle = Z\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \right)$$

$$= \cos\frac{\theta}{2}|0\rangle - e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$= \cos\frac{\theta}{2}|0\rangle + e^{i(\phi+\pi)}\sin\frac{\theta}{2}|1\rangle$$



$$\theta \longrightarrow \theta$$

$$\phi \longrightarrow \phi + \pi$$

$$Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$\boxed{X^2 = Y^2 = Z^2 = I}$$

Phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = |0\rangle\langle 0| + i\,|1\rangle\langle 1|$$

$$S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \neq S$$

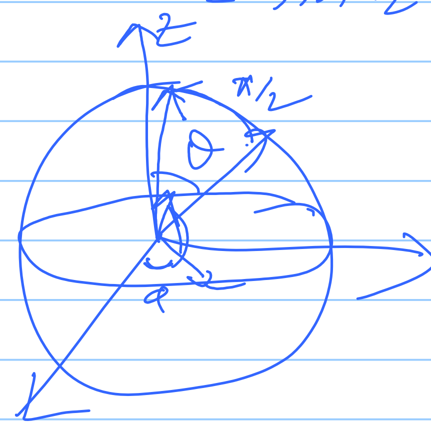
$$S^\dagger S = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = I$$

$$S|0\rangle = \left(|0\rangle\langle 0| + i\,|1\rangle\langle 1|\right)|0\rangle$$
$$= |0\rangle$$

$$S|1\rangle = i\,|1\rangle = \underline{e^{i\frac{\pi}{2}}}\,|1\rangle$$

$$S|\psi\rangle = S\left(\cos\tfrac{\theta}{2}|0\rangle + e^{i\phi}\sin\tfrac{\theta}{2}|1\rangle\right)$$
$$= \cos\tfrac{\theta}{2}|0\rangle + e^{i(\phi+\frac{\pi}{2})}\sin\tfrac{\theta}{2}|1\rangle$$

$$\boxed{S^4 = I}$$



Hadamard gate:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \boxed{H^\dagger H = I}$$

$$H^\dagger = H \Rightarrow \boxed{H^2 = I}$$

$$\underline{\underline{H|0\rangle}} = \underline{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\underline{\underline{\begin{bmatrix}1\\0\end{bmatrix}}} + \frac{1}{\sqrt{2}}\underline{\underline{\begin{pmatrix}0\\1\end{pmatrix}}}$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \underline{\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)}$$

$$\boxed{H|0\rangle = |+\rangle}$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

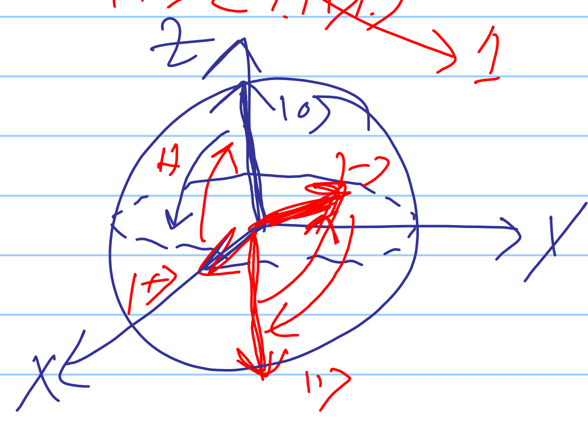
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\Big(|0\rangle\langle 0| + |0\rangle\langle 1|$$
$$+ |1\rangle\langle 0| - |1\rangle\langle 1|\Big)$$

$$\underline{\underline{H|1\rangle}} = \frac{1}{\sqrt{2}}\Big[|0\rangle\underline{\underline{\cancel{\langle 0|1\rangle}}}^{0} + |0\rangle\underline{\cancel{\langle 1|1\rangle}}^{1} + |1\rangle\cancel{\langle 0|1\rangle}^{0}$$
$$- |1\rangle\cancel{\langle 1|1\rangle}\Big)^{1}$$

$$= |-\rangle$$

$$H|-\rangle = H\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$



$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}\left(\cancel{|0\rangle} + |1\rangle - \cancel{|0\rangle} + |1\rangle\right)$$

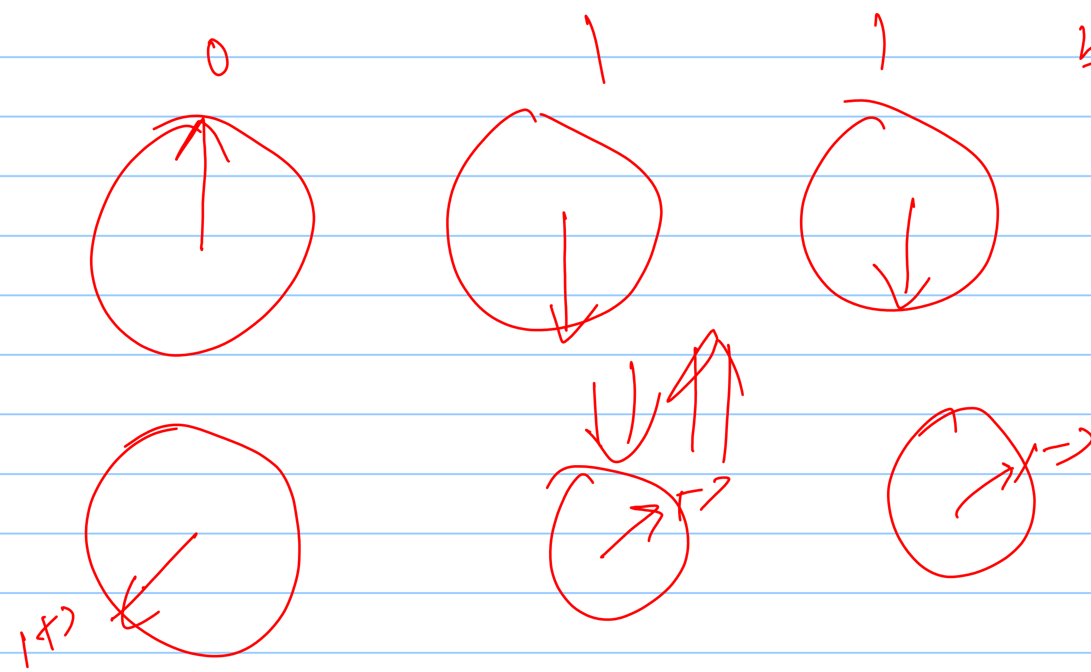
$$\boxed{H|-\rangle = |1\rangle}$$

$$H|+\rangle = |0\rangle$$

$$\{|0\rangle, |1\rangle\} \xrightarrow{H} \{|+\rangle, |-\rangle\}$$

$$\{|+\rangle, |-\rangle\} \xrightarrow{H} \{|0\rangle, |1\rangle\}$$

{comp basis}



Note Title 15/06/2021

Postulate 3: Measurement Postulate

$$|\psi'\rangle = M_m|\psi\rangle$$

$|\psi\rangle$

$$p(m) = \langle\psi'|\psi'\rangle$$

$$\boxed{|\psi\rangle = a|0\rangle + b|1\rangle}$$

$$M_m = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$p(0) = \langle\psi'|\psi'\rangle$$

$$|\psi'\rangle = \left(|0\rangle\langle 0|\right)\left(a|0\rangle + b|1\rangle\right)$$

$$= a|0\rangle$$

$$p(0) = \langle 0|a^* \, a|0\rangle = |a|^2\langle 0|0\rangle$$

$$\boxed{p(0) = |a|^2}$$

$$\boxed{p(1) = |b|^2}$$

$$p(0) + p(1) = 1$$

$$\boxed{I = |0\rangle\langle 0| + |1\rangle\langle 1|}$$

$$|\psi\rangle = e^{i\phi_1}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right]$$

$|\psi\rangle = a|0\rangle + b|1\rangle$