Solution Homework 1

PHY 414: Introduction to Quantum Computing

Problem 1: Using CNOT gate and single-qubit gates, construct a circuit to implement arbitrary controlled rotations around x axis $\mathrm{CR}_x(\theta)$. Implement this gate in qiskit and compare with their gate crx(theta, control qubit index, target qubit index).

Solution1: The circuit is illustrated in the figure below:

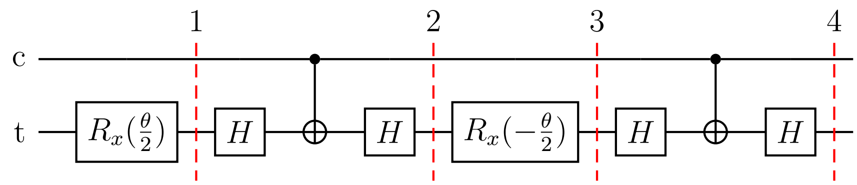

Figure 1: Circuit for controlled rotations around $x - axis$

Suppose our target qubit is in some arbitrary state: $|\psi\rangle$ as shown in figure:

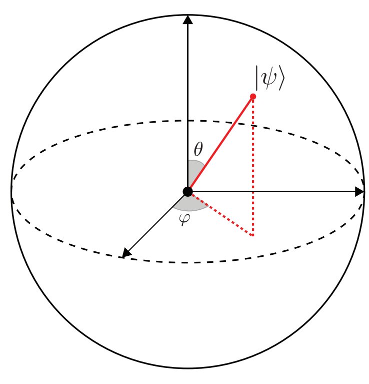

Figure 2: Initial state of Target qubit

The circuit do the following things (if the controlled qubit is 1).

1. It gives rotation by $\frac{\theta}{2}$ anti-clockwise around $x - axis$.

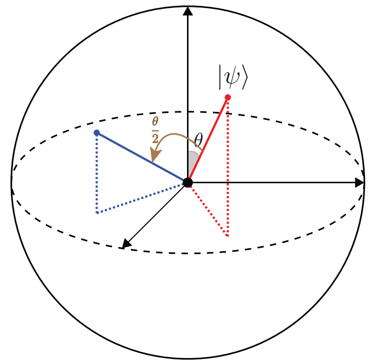

Figure 3: Step 01: Anti clockwise rotation around x-axis

2. It then gives 180^0 rotation around $z-axis$.

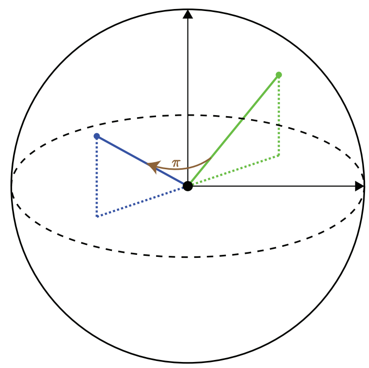

Figure 4: Step 02: 180^0 rotation around z-axis

3. It then gives $\dfrac{\theta}{2}$ clockwise around $x-axis$.

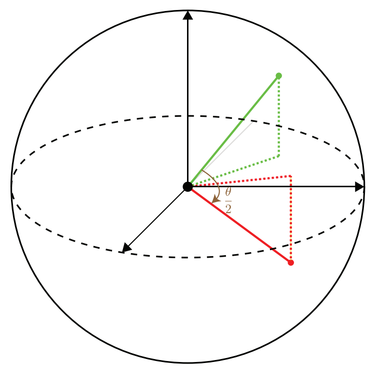

Figure 5: Step 03: Clockwise rotation around x-axis

4. At last, it gives 180^0 rotation around $z-axis$.

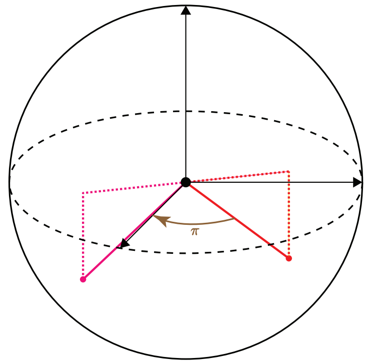

Figure 6: Step 04: 180^0 rotation around z-axis

Problem 2: In our class, we developed a teleportation circuit to transfer state of a qubit from S to B, if both the A and the B has $|\psi\rangle^{00}$ Bell state shared between them. Work out the circuit if both qubits had $|\psi\rangle^{01}$ Bell state instead of 00. Implement the circuit on Notebook and verify.

Solution: Sender has state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ which he is going to teleport such that both parties shares $|\psi\rangle^{01}$. The combines state initially is given by:

$$
\begin{aligned}
|\phi\rangle_S \otimes |\psi\rangle_{AB}^{01} &= [\alpha|0\rangle_S + \beta|1\rangle_S] \otimes \frac{1}{\sqrt{2}}[|01\rangle_{AB} + |10\rangle_{AB}] \\
&= \frac{1}{\sqrt{2}}[\alpha|001\rangle_{SAB} + \alpha|010\rangle_{SAB} + \beta|101\rangle_{SAB} + \beta|110\rangle_{SAB}]
\end{aligned}
$$

Writing each term twice (with 2 in denominator to balance) and subtracting the appropriate term from each term so that we end up with Bob having the state $|\phi\rangle$.

$$
\begin{aligned}
= \frac{1}{2\sqrt{2}}\Big[&\alpha|001\rangle + \alpha|111\rangle + \alpha|010\rangle + \alpha|100\rangle \\
&+ \alpha|001\rangle - \alpha|111\rangle + \alpha|010\rangle - \alpha|100\rangle \\
&+ \beta|101\rangle + \beta|011\rangle + \beta|110\rangle + \beta|000\rangle
\end{aligned}
$$

Factoring out the last qubit.

$$
\begin{aligned}
&+ \beta\,|101\rangle - \beta\,|011\rangle + \beta\,|110\rangle - \beta\,|000\rangle\Big] \\
&= \frac{1}{2\sqrt{2}}\Big[(|00\rangle + |11\rangle)(\alpha\,|1\rangle + \beta\,|0\rangle) \\
&\quad + (|01\rangle + |10\rangle)(\alpha\,|0\rangle + \beta\,|1\rangle) \\
&\quad + (|00\rangle - |11\rangle)(\alpha\,|1\rangle - \beta\,|0\rangle) \\
&\quad + (|01\rangle - |10\rangle)(\alpha\,|0\rangle - |1\rangle)\Big]
\end{aligned}
$$

We know that:

$$
|\psi\rangle^{00} = \frac{|00\rangle + |11\rangle}{\sqrt{2}},\ |\psi\rangle^{01} = \frac{|01\rangle + |10\rangle}{\sqrt{2}},\ |\psi\rangle^{10} = \frac{|00\rangle - |11\rangle}{\sqrt{2}},\ |\psi\rangle^{11} = \frac{|01\rangle + |10\rangle}{\sqrt{2}}
$$

And we see that:

$$\text{(i)}\ \alpha|1\rangle + \beta|0\rangle = \hat{X}|\phi\rangle$$

$$\text{(ii)}\ \alpha|0\rangle + \beta|1\rangle = \hat{I}|\phi\rangle$$

$$\text{(iii)}\ \alpha|1\rangle\,\beta|0\rangle = \hat{X}\hat{Z}|\phi\rangle$$

$$\text{(iv)}\ \alpha|0\rangle - \beta|1\rangle = \hat{Z}|\phi\rangle$$

$$|\phi\rangle_S|\psi\rangle_{AB}^{01} = \frac{1}{2}\left[|\psi\rangle_{SA}^{00}\otimes\hat{X}|\phi\rangle_B + |\psi\rangle_{SA}^{01}\otimes\hat{I}|\phi\rangle_B + |\psi\rangle_{SA}^{11}\otimes\hat{X}\hat{Z}|\phi\rangle_B + |\psi\rangle_{SA}^{10}\otimes\hat{Z}|\phi\rangle_B\right]$$

Thus if we measure qubits of Sender and Alice in Bell's Basis and get:

i) $|\psi\rangle^{00} \rightarrow$ Bob will apply $\hat{X}$ to his state, to retrieve $|\phi\rangle$

ii) $|\psi\rangle^{01} \rightarrow$ Bob will apply $\hat{I}$ to his state, to retrieve $|\phi\rangle$ or in other words, he will do nothing.

iii) $|\psi\rangle^{11} \rightarrow$ Bob will apply $\hat{X}\hat{Z}$ to his state, to retrieve $|\phi\rangle$

iv) $|\psi\rangle^{10} \rightarrow$ Bob will apply $\hat{Z}$ to his state to, retrieve $|\phi\rangle$

The circuit for teleportation is following:

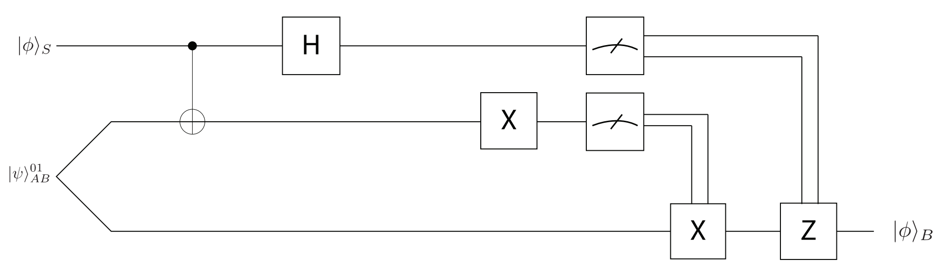

Figure 7: Circuit for Teleportation

Problem 3: Using phase kickback, construct a circuit of $3+1$ qubits that gives a negative sign at the output multiplied with the same input if the input of first three qubits is $000, 010, 111, 101$ and gives the same output as input if the input is $001, 011, 100, 110$. Implement the circuit in qiskit and verify.

Solution 3: Since we need negative phase when the the first and last qubit is of the same sign, the oracle given in below figure will do the job.

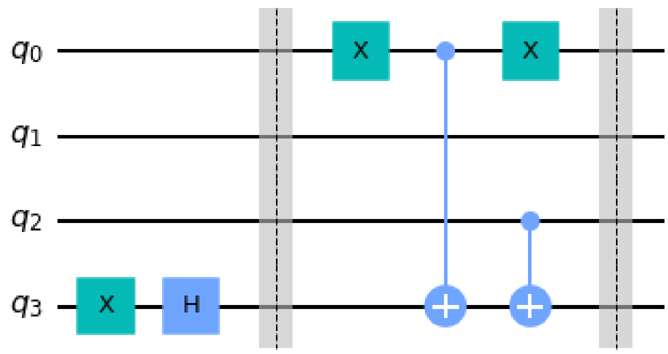

Figure 8: Oracle for Q. 3.