Solution Homework 2

PHY 414: Introduction to Quantum Computing

Problem 1: The 'Qiskit' library does not contain a built-in method for implementing the controlled S-gate (as seen in Activity 3). You are required to make the controlled-S gate by using other available gates in the library. This can be done by using only controlled X-gate(s) and Z rotation-gate(s).

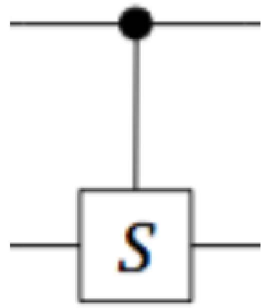

Figure 1: Controlled S gate

The operator form of the controlled S gate is as follow:

$$S = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + \exp(i\pi/2)\,|11\rangle\langle 11| \tag{1}$$

Solution 1: Control S gate is a 90 deg rotation rotation around z axis controlled by control bit. It can easily be accomplished by a 45 deg rotation around z axis and CNOT gate, as shown in Fig. 1.

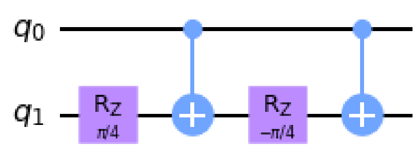

Figure 2: Circuit for Controlled-S gate

Problem 2: In the class, we showed that Deutsch–Josza algorithm states that the function in the phase oracle is constant if the output is state $|0\rangle^{\otimes n}$ and the function is balanced if the output is any other state. This was when the input was initialized to state $|0\rangle^{\otimes n}$. For this problem, assume that the initial state of all qubits is $|1\rangle^{\otimes n}$. Now, prove that the function is constant if the output is $|1\rangle^{\otimes n}$ and is balanced if any other state is output. Check it in qiskit as well.

Solution 2: Algorithm is to prepare a bunch of qubits in $|1\rangle$ states and then we perform hadamard gate to each qubit independently in order to create superposition states. After that, we apply phase oracle U_f to all the superposition states. At last, Hadamard gates are applied to the output of oracle followed by measuting each qubit. Based on the measurement results, we would be able to tell if the function is balanced or constant. Circuit for this algorithm is illustrated in figure below:

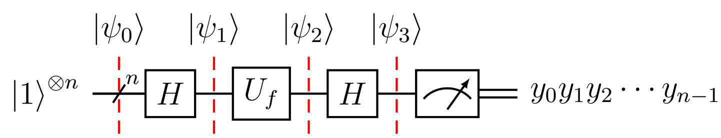

Figure 3: Deutsch-Jozsa Algorithm

Now, let's workout each steps of the algorithm:

1. Prepare a $n-$qubit register initialized to 0.

$$|\psi_0\rangle = |1\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle = |1\rangle^{\otimes n} \tag{2}$$

2. Apply Hadamard gate to each qubit.

$$\begin{aligned} |\psi_1\rangle &= H^{\otimes n} |\psi_0\rangle \\ &= H^{\otimes n} |1\rangle^{\otimes n} \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{k.111\cdots 11} |k\rangle \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{k} |k\rangle \end{aligned} \tag{3}$$

We know that,

$$
\begin{aligned}
H|y_0\rangle &= \frac{|0\rangle + (-1)^{y_0}|1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \sum_{k_0=0}^{1} (-1)^{k_0 y_0} |k_0\rangle
\end{aligned}
$$

Thus for n-qubit,

$$
\begin{aligned}
H^{\otimes n}|y\rangle &= \frac{1}{\sqrt{2^n}} \sum_{k_0}^{1} \sum_{k_1}^{1} \cdots \sum_{k_{n-1}}^{1} (-1)^{k_{n-1}y_{n-1} \oplus \cdots \oplus k_0 y_0} |k_{n-1}k_{n-2} \cdots k_1 k_0\rangle \\
H^{\otimes n}|y\rangle &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{\vec{k}\cdot\vec{y}} |k\rangle
\end{aligned}
$$

But, $y \in \{000\cdots00\ ,\ 000\cdots01\ , \cdots ,\ 111\cdots11\}$.

$$
\begin{aligned}
\Rightarrow\ & 000\cdots00 \ \cdot\ 111\cdots11 = 0 \\
& 000\cdots01 \ \cdot\ 111\cdots11 = 1 \\
& \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
& 111\cdots11 \ \cdot\ 111\cdots11 = N-1 \\
\Rightarrow\ & \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{k\cdot 111\cdots111} |k\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{k} |k\rangle
\end{aligned} \tag{4}
$$

3. Apply the phase oracle to the state $|\psi_1\rangle$.

$$
\begin{aligned}
|\psi_2\rangle = U_f|\psi_1\rangle &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{k} U_f |k\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{f(x)+k} |k\rangle
\end{aligned} \tag{5}
$$

4. Apply hadamard gate to the state $|\psi_2\rangle$.

$$
\begin{aligned}
|\psi_3\rangle &= H^{\otimes n} |\psi_2\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{f(x)+k} H^{\otimes n} |k\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^{f(x)+k} \left(\sum_{l=0}^{N-1} \frac{1}{\sqrt{N}} (-1)^{l\cdot k} |l\rangle \right) \\
&= \sum_{l=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} (-1)^{k+f(x)+l\cdot k} \right) |l\rangle \\
&= \sum_{l=0}^{N-1} C_l |l\rangle \\
&= C_0 |0\rangle + C_1 |1\rangle + \cdots + C_{N-1} |N-1\rangle
\end{aligned} \tag{6}
$$

The co-efficient of the state $|N-1\rangle = |111\cdots 11\rangle$ is:

$$
C_{N-1} = \frac{1}{N} \sum_{k=0}^{N-1} (-1)^{k+f(x)} (-1)^{k\cdot 111\cdots 11}
$$

But from Eq.4, we know that $\sum_{k=0}^{N-1} (-1)^{k\cdot 111\cdots 11} = \sum_{k=0}^{N-1} (-1)^k$. Thus:

$$
\begin{aligned}
C_{N-1} &= \frac{1}{N} \sum_{k=0}^{N-1} (-1)^{2k+f(x)} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} (-1)^{2k} (-1)^{f(x)} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} (-1)^{f(x)} \quad \because (-1)^{2k} = 1
\end{aligned}
$$

Therefore, probability of finding the state $|111\cdots 11\rangle$ is:

$$
\begin{aligned}
\Pr(111\cdots 11) &= |C_{N-1}|^2 \\
&= \left| \frac{1}{N} \sum_{k=0}^{N-1} (-1)^{f(x)} \right|^2
\end{aligned}
$$

Clearly,

$$
\Pr(111\cdots 11) = \begin{cases} 1 & \text{If } f(x) \text{ is constant} \\ 0 & \text{If } f(x) \text{ is balanced} \end{cases}
$$

Thus, if we measure the state $|111\cdots 11\rangle$ with 100% probability, it means that function is constant otherwise balanced.

Problem 3: Implement Berstein–Vazirani circuit for secret bit string $s = 1011$ and run your code on two quantum computers of IBM for 1000 and 5000 shots (on each of these quantum computers). By comparing your results with the simulator's results, can you decide which of these quantum computers has less noise?

Solution 3: Generally, more iterations of a quantum code should give better estimates of the output quantum state. However, if the quantum computer is too noisy, more runs are not guaranteed to give better estimates. Your answer will depend on your choice of quantum computer.