Midterm Solution

PHY 414: Introduction to Quantum Computing

Problem 1: Answer the following short questions:

1. What do you think is the major source of the power of quantum computers?
2. Why Bernstein–Vazirani algorithm is important in showing the power of quantum computers more so than the DJ algorithm?
3. What is one main application of teleportation on a quantum computer?

Solution:

1. Rather than bits which can be either 0 or 1, quantum computers use qubits that can be 0 and 1 simultaneously, i.e., the qubit can be in superposition states. The ability to process data in superposition is the major source of the power of quantum computers.
2. Deutsch-Jozsa Algorithm aimed to determine whether a function is constant or balanced, which can also be solved by probabilistic classical computer in a polynomial number of iterations. Even though quantum computer gives exponential speedup over classical solutions, the speedup over the probabilistic classical computer is not much. But Bernstein-Vazirani Algorithm gives us a clear cut speed up over even a probabilistic classical computer because a probabilistic classical computer cannot reveal the hidden bit string in few queries.
3. Quantum teleportation enables the transfer of quantum information in the form of quantum states from one place to another. On quantum computers where we have physical qubits, the hardware allows entangling operations only with physically adjoining qubits. To perform an entangling operation between the qubits far from each other, we use teleportation protocol to transfer qubit from one place to another and then perform the required entangling operations.

Problem 2: Construct a **(3+1)** or **(3+2)**-qubit phase oracle that transforms 3-qubit input $|x\rangle$ to $(-1)^{f(x)}|x\rangle$ where $f(x) = 1$ when the 3-qubit $|x\rangle$ is binary representation of a prime number (1, 2, 3, 5, or 7) and $f(x) = 0$ when $|x\rangle$ is binary representation of **0, 4,** or **6**. Implement this oracle in qiskit and evolve each of the eight input states and plot the output on Bloch sphere and Qsphere for all qubits (including the ancillary qubits).

Solution:

Method: 1

We need to implement $f(x)$ such that:

$$f(x) = \begin{cases} 1 & x: \ 1, 2, 3, 5, 7 \\ 0 & x: \ 0, 4, 6 \end{cases} \tag{1}$$

So our truth table should looks like:

Input	Output
$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$-\|001\rangle$
$\|010\rangle$	$-\|010\rangle$
$\|100\rangle$	$\|100\rangle$
$\|011\rangle$	$-\|011\rangle$
$\|101\rangle$	$-\|101\rangle$
$\|110\rangle$	$\|110\rangle$
$\|111\rangle$	$-\|111\rangle$

Table 1: Truth table for the required oracle

We want to add phase to the qubits: 001, 010, 011, 101 and 111. The circuit that will do the job is given below:

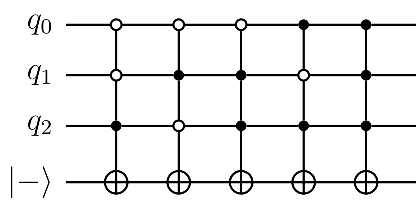
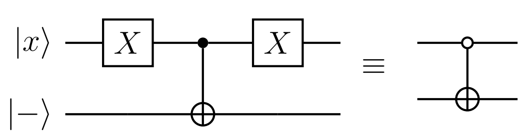
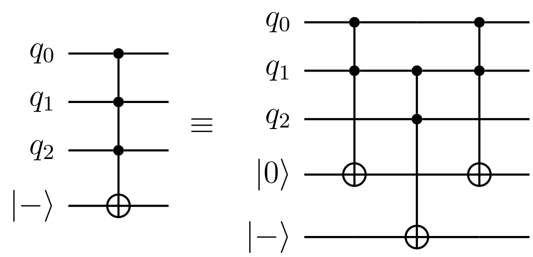


Figure 1: Circuit for the required oracle

where:

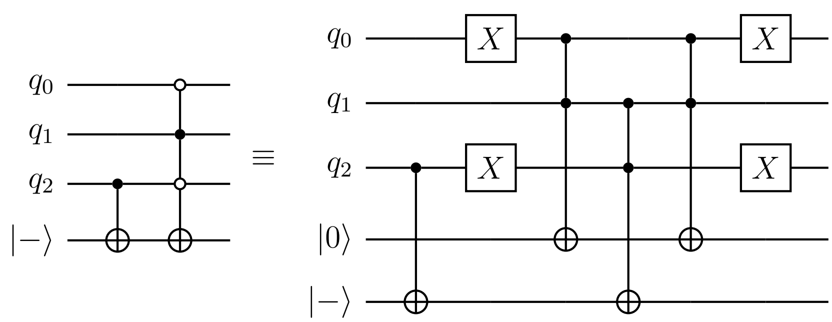


and,



Method: 2

There is another way of building circuit that will do the job. If we see in the truth table, phase is added to only those states whose last qubit is 1 with an exception of 010. So we will build circuit that perform controlled operation with respect to the last qubit. After that, we will take care of the state 010. The circuit will look like:



Note: Order of qubits is from left to right.

Problem 3: If the input of the Grover search algorithm is initialized to state $|111...1\rangle$ and the Diffuser is defined as $V = H^{\otimes n}(I - 2\,|1111...1\rangle\,\langle 111...1|)H^{\otimes n}$, will the algorithm still be able to search the solution the same way as it worked for the case when Grover algorithm was initialized to $|00..0\rangle$ state? Prove or disprove by working out the derivation of an algorithm. You don't have to do the complete derivation, and you can use the results that are already derived in the class. Only that much is required, which will prove or disprove the use of the new scheme.

Solution: Given that initial state is:

$$|\psi_{inp}\rangle = |111\cdots 11\rangle$$

And the diffuser operator is given by:

$$V = H^{\otimes n}\Big(\mathbb{I} - 2\,|111\cdots 11\rangle\langle 111\cdots 11|\,\Big)H^{\otimes n}$$

Let's start by circuit for grover algorithm.

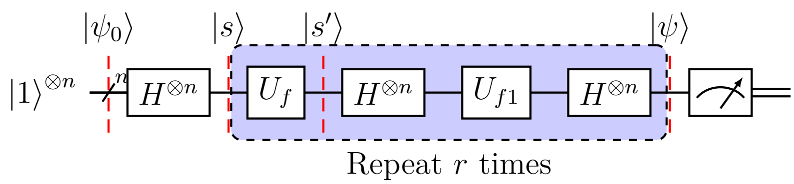


Figure 2: Circuit for Grover Search Algorithm

1. The state $|s\rangle$ is:

$$\begin{aligned}
|s\rangle &= H^{\otimes n}\,|111\cdots 11\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{\vec{x}\cdot 1}\,|x\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{x}\,|x\rangle \\
&= \frac{1}{\sqrt{N}}\Big(\,|0\rangle - |1\rangle + |2\rangle - \cdots + (-1)^{w}\,|w\rangle + \cdots + (-1)^{N-1}\,|N-1\rangle\,\Big) \\
&= \frac{1}{\sqrt{N}}\sum_{x\neq w}(-1)^{x}\,|x\rangle + \frac{1}{\sqrt{N}}\,|w'\rangle \quad : |w'\rangle = (-1)^{w}\,|w\rangle \\
&= \sqrt{\frac{N-1}{N}}\frac{1}{\sqrt{N-1}}\sum_{x\neq w}(-1)^{x}\,|x\rangle + \frac{1}{\sqrt{N}}\,|w'\rangle \\
&= \sqrt{\frac{N-1}{N}}\,|w'^{\perp}\rangle + \frac{1}{\sqrt{N}}\,|w'\rangle
\end{aligned}$$

Take $\cos(\alpha) = \sqrt{\frac{N-1}{N}}$ and $\sin(\alpha) = \frac{1}{\sqrt{N}}$. So we can write:

$$|s\rangle = \cos(\alpha)\,|w'^{\perp}\rangle + \sin(\alpha)\,|s'\rangle$$

2. Now, applly phase oracle to the $|s\rangle$ state:

$$U_f\,|s\rangle = |s'\rangle = \cos(\alpha)\,|w'^{\perp}\rangle - \sin(\alpha)\,|s'\rangle$$

3. Now, the diffuser can be written as:

$$\begin{aligned} V &= H^{\otimes n}\Big(\mathbb{I} - 2\,|1\rangle\!\langle 1|\Big)H^{\otimes n} \\ &= H^{\otimes n}H^{\otimes n} - 2H^{\otimes n}\,|1\rangle\,\langle 1|\,H^{\otimes n} \\ &= \mathbb{I} - 2\,|s\rangle\!\langle s| \end{aligned}$$

Apply the diffuser on $|s'\rangle$:

$$\begin{aligned} V\,|s'\rangle &= \Big(\mathbb{I} - 2\,|s\rangle\!\langle s|\Big)\,|s'\rangle \\ &= \cos(3\alpha)\,|w'^{\perp}\rangle + \sin(3\alpha)\,|w'\rangle \end{aligned}$$

So we see that if input states are initialized to $|111\cdots 11\rangle$, the algorithm is still able to search the solution.